

**SELF-DUALITY IN DIMENSIONS $2n > 4$: EQUIVALENCE OF VARIOUS
DEFINITIONS AND AN UPPER BOUND FOR p_2**

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Abstract

We show that the self-duality defined in [Trautman, Int.J.Theor.Phys., **16**, 561 (1977)] is equivalent to strong self-duality defined in [Bilge, Dereli and Kocak, Lett.Math.Phys., **36**, 301-309, (1996)] and we obtain an upper bound on $\int (p_2)^n$, where p_2 is the second Pontrjagin class of an $SO(N)$ bundle over a $8n$ dimensional manifold.

1. Introduction.

The self-duality of a 2-form in four dimensions is defined to be the Hodge duality [1]. The search for a meaningful definition of self-duality in higher dimensions as a map on 2-forms, has been studied by various authors [2-7]. A definition of self-duality based on an eigenvalue criterion has been recently proposed in [8] where an overview of previous results is also given. According to this definition, the “strongly self-dual” 2-forms in $2n$ dimensions form an $n^2 - n + 1$ dimensional submanifold \mathcal{S}_{2n} of the skew-symmetric $2n \times 2n$ matrices, determined by the condition that the absolute values of the eigenvalues coincide. It was also shown that this definition is equivalent to the self-duality condition used by Grossman et.al. [6], namely $F \wedge F$ be self-dual in the Hodge sense. Here, in Section 2 we show that the self-duality condition given by Trautman [3] $\omega^{n-1} = k * \omega$, where $*$ denotes the Hodge dual, is also equivalent to “strong self-duality”.

In [8], it has also been shown that for an $SO(N)$ bundle, the strongly self-dual forms saturate an upper bound for the integral of the first Pontrjagin class p_1 . Here we obtain an upper bound for the integral of the second Pontrjagin class p_2 for an $SO(N)$ bundle, which is again saturated for strongly self-dual forms. This upper bound is however gauge independent only under an “orthonormality” assumption (Eq.3.4).

2. Equivalence of various definitions of self-duality in $2n$ dimensions.

2.1. Preliminaries.

Let M be a $2n$ dimensional manifold, E be a vector bundle over M with standard fiber R^N and structure group G , and let \mathbf{g} be a linear representation of the Lie algebra of G . If A is the \mathbf{g} valued local connection 1-form, the local curvature 2-form is defined as $F = dA - A \wedge A$ and the Bianchi identities are $dF + F \wedge A - A \wedge F = 0$. The local curvature 2-form depends on the trivialization of the bundle, but its invariant polynomials σ_i defined by $\det(I + tF) = \sum_{i=0}^n \sigma_i t^i$ are invariant of the local trivialization. The σ_i ’s are closed $2i$ -forms, defining deRham cohomology classes in H^{2i} , proportional to the Chern classes \mathbf{c}_i (or the Pontrjagin classes $\mathbf{p}_{i/2}$) of the bundle E . In addition for an $SO(N)$ bundle, the square root of the determinant of F (which is a ring element) defines the Euler class χ [9]. The σ_i ’s can also be written as linear combinations of $\text{tr} F^i$

[10], where F^i means the product of the matrix F with itself i times, with the wedge multiplication of the entries. In order to avoid proportionality constants, we will work with the quantities σ_i 's instead of the Chern or Pontrjagin classes. In the following we consider real $SO(N)$ bundles, and we recall that all principal minors of F are skew-symmetric matrices, hence their determinants are perfect squares, and the σ_{2i} 's are sums of squares of i -fold products of the entries of F .

We shall use the notation, (a, b) where a, b are forms on M to represent the inner product in on the exterior algebra ΛM_x , $x \in M$, while $\langle A, B \rangle$ where A, B are Lie algebra valued forms will denote the inner product in $\Lambda M_x \otimes E_x$, $x \in M$, hence both expressions are functions defined on the manifold.

2.2. Equivalence of strong self-duality and Trautman's definition.

Let ω_{ij} be the components of a 2-form in $2n$ dimensions with respect to some local orthonormal basis. In the following, we shall denote the 2-form ω and the skew-symmetric matrix consisting of its components with respect to some orthonormal basis by the same symbol. Let $\pm i\lambda_k$, $k = 1 \dots n$ be the eigenvalues of the skew-symmetric matrix ω . The invariant polynomials s_{2i} , of ω can be expressed in terms of the elementary symmetric functions of the λ_i^2 's. The inner products (ω^i, ω^i) and the s_{2i} 's are related as follows.

$$\begin{aligned}
(\omega, \omega) &= s_2 = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2, \\
\frac{1}{(2!)^2} (\omega^2, \omega^2) &= s_4 = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \dots + \lambda_{n-1}^2 \lambda_n^2 \\
\frac{1}{(3!)^2} (\omega^3, \omega^3) &= s_6 = \lambda_1^2 \lambda_2^2 \lambda_3^2 + \lambda_1^2 \lambda_2^2 \lambda_4^2 + \dots + \lambda_{n-2}^2 \lambda_{n-1}^2 \lambda_n^2 \\
&\dots \dots \\
\frac{1}{(n!)^2} (\omega^n, \omega^n) &= \frac{1}{(n!)^2} |\ast \omega^n|^2 = s_{2n} = \lambda_1^2 \lambda_2^2 \dots \lambda_n^2
\end{aligned} \tag{2.1}$$

We define the weighted elementary symmetric polynomials by $q_i = s_{2i} / \binom{n}{i}$. We have the inequalities [11]

$$q_1 \geq q_2^{1/2} \geq q_3^{1/3} \geq \dots \geq q_n^{1/n}, \quad q_{r-1} q_{r+1} \leq q_r^2, \quad 1 \leq r < n \tag{2.2}$$

and the equalities hold iff all the λ_i 's are equal. In [8] we have adopted this case as a definition.

Definition 2.1 Let ω be a 2-form in $2n$ dimensions, $\pm i\lambda_k$, $k = 1, \dots, n$ be its eigenvalues and η be the square root of the determinant of ω , with a fixed choice of sign. Then ω is called *strongly self-dual* (*strongly anti self-dual*) if $|\lambda_1| = |\lambda_2| = \dots = |\lambda_n|$, and $\eta > 0$ ($\eta < 0$).

The strong self-duality condition is equivalent to the matrix equation $\omega^2 + \lambda^2 I = 0$, where I is the identity matrix, and $\lambda^2 = \frac{1}{2n} \text{tr} \omega^2$. This definition gives quadratic equations for the ω_{ij} 's, hence the strong self-duality condition determines a nonlinear set. In four dimensions, the strong self-duality coincides with usual Hodge duality, more precisely, the matrices satisfying $\omega^2 + \lambda I = 0$ consist of the union of the usual self-dual and anti self-dual forms. In higher dimensions the set \mathcal{S}_{2n} is an $n^2 - n + 1$ dimensional submanifold [12].

We recall the inequalities

$$(\omega, \eta)^2 \leq (\omega, \omega)(\eta, \eta), \quad 2(\omega, \eta) \leq (\omega, \omega) + (\eta, \eta). \quad (2.3)$$

From the (2.1) and (2.3) we obtain the following Lemma which generalizes the results of [8] to arbitrary dimensions.

Lemma 2.2. *Let ω be a 2-form in $2n$ dimensions. Then*

$$(n-1)(\omega, \omega)^2 - \frac{n}{2}(\omega^2, \omega^2) \geq 0 \quad (2.4)$$

$$(\omega^{n/2}, \omega^{n/2}) \geq *\omega^n, \quad (2.5)$$

and equality holds if and only if all eigenvalues of ω are equal.

Proof. To obtain Eq.(2.4) we use,

$$\begin{aligned} q_1^2 &\geq q_2 \\ \frac{1}{n^2} s_2^2 &\geq \frac{2}{n(n-1)} s_4 \\ \frac{1}{n^2} (\omega, \omega)^2 &\geq \frac{2}{n(n-1)} \frac{1}{4} (\omega^2, \omega^2) \end{aligned}$$

which gives the desired result. Similarly using

$$\begin{aligned} q_{n/2}^2 &\geq q_n \\ \left(\frac{(n/2)!(n/2)!}{n!} s_{n/2} \right)^2 &\geq s_{2n} \\ \left(\frac{1}{n!} (\omega^{n/2}, \omega^{n/2}) \right)^2 &\geq \frac{1}{(n!)^2} |*\omega^n|^2 \end{aligned}$$

we obtain (Eq.(2.5)). e.o.p.

From Lemma 2.2, we immediately have

Corollary 2.3. *The 2-form ω is strongly self-dual iff $\omega^{n/2}$ is self-dual in the Hodge sense.*

We will now show that this condition is also equivalent to the self-duality definition used by Trautman [3].

Proposition 2.4. *Let ω be a 2-form in $2n$ dimensions. Then*

$$\omega^{n-1} = k * \omega \quad (2.6)$$

where k is a constant, if and only if ω is strongly self-dual and $k = \frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1}$.

Proof. Multiplying (2.6) with ω and taking Hodge duals, we obtain, $*\omega^n = k(\omega, \omega)$. Since $(\omega, \omega) = s_2 = nq_1$ and $|*\omega^n| = n!s_{2n}^{1/2} = n!q_n$, we obtain $k = (n-1)!q_n^{1/2}/q_1$. Then taking inner products of both sides of (2.6) with themselves, we obtain $(\omega^{n-1}, \omega^{n-1}) = k^2(*\omega, *\omega) = k^2(\omega, \omega)$. Substituting the value of k obtained above, and using $(\omega^{n-1}, \omega^{n-1}) = ((n-1)!)^2 nq_{n-1}$, we obtain $q_n = q_{n-1}q_1$. But since $q_1 \geq q_n^{1/n}$, we have $q_n \geq q_{n-1}q_n^{1/n}$, which leads to $q_n^{n-1} \geq q_{n-1}$. This is just the reverse of the inequality in (2.2), hence equality must hold, and all eigenvalues of ω are equal in absolute value. Thus ω is strongly self-dual and it can also be seen that $k = \frac{n!}{n^{n/2}}(\omega, \omega)^{\frac{n}{2}-1}$. e.o.p.

Applying Lemma 2.2 to the 2-forms $\omega \pm \eta$ we obtain

Corollary 2.5. *Let ω and η be 2-forms in $2n$ dimensions. Then*

$$\begin{aligned} 4(\omega\eta, \omega\eta) \leq & 4\frac{2(n-1)}{n}(\omega, \eta)^2 + \left[\frac{2(n-1)}{n}(\omega, \omega)^2 - (\omega^2, \omega^2) \right] + \left[\frac{2(n-1)}{n}(\eta, \eta)^2 - (\eta^2, \eta^2) \right] \\ & + 2 \left[\frac{2(n-1)}{n}(\omega, \omega)(\eta, \eta) - (\omega^2, \eta^2) \right] \end{aligned}$$

The inequality is saturated when $\omega = \eta$ and it is strongly self-dual. This corollary will be used in the next section.

3. An upper bound bound for σ_4 .

Consider an $SO(N)$ valued curvature 2-form $F = (F_{ab})$. Then

$$\langle F, F \rangle = 2 \sum (F_{ab}, F_{ab}), \quad (3.1a)$$

$$\sigma_2 = \sum F_{ab}^2 \quad (3.1b),$$

$$\sigma_4 = \sum [F_{ab}F_{cd} - F_{ac}F_{bd} + F_{ad}F_{bc}]^2, \quad (3.1c)$$

where the summation goes over the indices such that $b > a$, $d > c$, and in the last summation the pairs $\{a, b\}$ and $\{c, d\}$ are distinct with $c > a$. We recall the integrals $\int (\sigma_2)^k$ (over a $4k$ manifold) and $\int (\sigma_4)^k$ (over an $8k$ manifold) are independent of the bundle trivialization *and* of the representatives of the cohomology classes, hence they are invariants of the bundle, while, $\int \langle F, F \rangle^k$ and $\int (\sigma, \sigma)^k$ are only independent of the bundle trivializations. Thus our aim is to obtain upper bounds for the bundle invariant quantities in terms of latter quantities, which can be taught of physical fields, and to characterize the conditions under which these inequalities are saturated.

In [8] we have obtained an upper bound for $\int \sigma_2^2$ for an $SO(N)$ bundle over an eight dimensional manifold. This result can be generalized immediately to $SO(N)$ bundles over $2n$ dimensional manifolds.

Proposition 3.1. *Let F be the curvature 2-form of an $SO(N)$ bundle over a $2n$ dimensional manifold, Then*

$$\langle F, F \rangle^2 \geq \frac{1}{4} \frac{2(n-1)}{n} (\sigma_2, \sigma_2) \geq \frac{1}{4} \frac{2(n-1)}{n} * \sigma_2^2. \quad (3.2)$$

The conditions under which the inequalities are saturated depend both on the relations among the F_{ab} 's, and on the properties of each F_{ab} , i.e., inequalities of type (2.3) and (2.4) should be both saturated. For example, (3.2) is saturated when $F = \omega F_o$, where F_o is a constant matrix, and ω is a strongly self-dual 2-form. By raising both sides of the inequality in (3.2) to power k and integrating, we obtain upper bounds for the integral of p_1^k on a $4k$ dimensional manifold.

We can give estimates on σ_4 for an $SO(N)$ bundle using Corollary 2.5. To illustrate the proof, we will first work for an $SO(4)$ bundle, then generalize the result to arbitrary dimensions.

Proposition 3.2. *Let F be the curvature 2-form of an $SO(4)$ bundle over an eight dimensional manifold. Then*

$$| * \sigma_4 | \leq \frac{9}{2} \Phi + \frac{9}{32} \langle F, F \rangle^2 - \frac{3}{4} (\sigma_2, \sigma_2) \quad (3.3)$$

where $\Phi = (F_{12}, F_{34})^2 + (F_{13}, F_{24})^2 + (F_{14}, F_{23})^2$.

Proof. We explicitly write

$$\langle F, F \rangle = 2 [(F_{12}, F_{12}) + (F_{13}, F_{13}) + (F_{14}, F_{14}) + (F_{23}, F_{23}) + (F_{24}, F_{24}) + (F_{34}, F_{34})],$$

$$\sigma_2 = F_{12}^2 + F_{13}^2 + F_{14}^2 + F_{23}^2 + F_{24}^2 + F_{34}^2,$$

$$\sigma_4 = [F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}]^2.$$

Then using the inequality $*\eta^2 < (\eta, \eta)$ for 4-forms, we have

$$|*\sigma_4| \leq (F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}, F_{12}F_{34} - F_{13}F_{24} + F_{14}F_{23}),$$

which gives using (2.3b)

$$|*\sigma_4| \leq 3 [(F_{12}F_{34}, F_{12}F_{34}) + (F_{13}F_{24}, F_{13}F_{24}) + (F_{14}F_{23}, F_{14}F_{23})]$$

Now we can use Corollary 2.5 to write the right hand side in terms of $\langle F, F \rangle$ and (σ_2, σ_2) . However we need to add terms such as $3(F_{12}, F_{12})(F_{13}, F_{13}) - 2(F_{12}^2, F_{13}^2)$ (which are all positive quantities) in order to obtain $\langle F, F \rangle$ and (σ_2, σ_2) on the right hand side. Collecting terms we obtain (3.3). e.o.p.

The inequality is saturated when $F_{12} = F_{34}$, $F_{13} = -F_{24}$, $F_{14} = F_{23}$, and each of these 2-forms are strongly self-dual. The quantity Φ is not invariant under bundle automorphisms in general, however it can be seen that the condition $\Phi = 0$ is invariant if

$$(F_{ij}, F_{kl}) = \delta_{ik}\delta_{jl} \quad (3.4).$$

It is possible to obtain a bound for Φ in terms of $\langle F, F \rangle$, but then the inequality (3.3) is not saturated.

The Proposition can be generalized to arbitrary dimensions by counting arguments.

Corollary 3.3. *Let F be the curvature 2-form of an $SO(N)$ bundle over a $2n$ dimensional manifold. Then*

$$|*\sigma_4| \leq \frac{3}{4} \left[6\Phi + \frac{3}{8} \binom{N-2}{2} \langle F, F \rangle^2 - \binom{N-2}{2} (\sigma_2, \sigma_2) \right] \quad (3.5)$$

where $\Phi = \sum(F_{ij}, F_{kl})$, where the summation extends over distinct pairs of indices $\{i, j\}$ and $\{k, l\}$.

By raising both sides of (3.4) to power k and integrating we obtain upper bounds for $\int (p_2)^k$ over an $8k$ dimensional manifold. The right hand side of Eq.(3.5) is gauge invariant only if $\Phi = 0$ and an analogue of Eq.(3.4) holds.

4. Conclusion.

The equivalence of strong self-duality and various other definitions of self-duality in dimensions higher than four, and the saturation of certain topological lower bounds suggests that strongly self-dual forms share many important properties of self-dual 2-forms in four dimensions. There are however certain features that are not matched completely.

First of all the strongly self-dual 2-forms do not form a linear space. This situation can be remedied by restricting the notion of self-duality to linear submanifolds [12]. However the maximal dimension of such linear submanifolds in dimensions $2n$ is equal to the number of linearly independent vector fields on S^{2n-1} . In eight dimensions those linear submanifolds include a representation of octonions in R^8 and the self-duality equations of Corrigan et al [2]. However in dimensions for example $2(2a + 1)$ these linear subspaces are one dimensional, and the self-duality equations obtained in [2] cannot be represented in this framework. This suggests that the notion of strong self-duality may be weakened to comprise a richer structure of self-duality equations, at the price of relaxing the saturation of topological lower bounds.

The second problem is that, the self-duality equations obtained by selecting linear subspaces of strongly self-dual forms are overdetermined. Thus the existence of solutions are not guaranteed in general (one solution is given in [6]). We remark however that in eight dimensions, the orthogonal complements of the linear subspaces of strongly self-dual forms lead to elliptic equations, and this suggest that in arbitrary dimensions, these complements may be more interesting from the point of view of resulting system of partial differential equations.

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